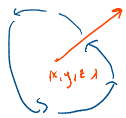
Math 53, Discussion 105

Friday, April 17, 2020 9:42 AM

16.5 (Curl and Divergence)
16.6 (Parametric Surfaces and Their Areas)



curl $F(x,y,z)$

$\frac{d}{dx}$, $f\frac{d}{dx}g + g\frac{d}{dx}f$

$u:\mathbb{R}^3\to\mathbb{R}^3$ $u\cdot\nabla$ $\neq$ $\nabla\cdot u$

$\boxed{(u\cdot\nabla)u}$ $-(\nabla\cdot u)\,u$

$(u\cdot\nabla)u$

$(\langle u_1,u_2,u_3\rangle\cdot\langle\partial_x,\partial_y,\partial_z\rangle)u$

$(u_1\partial_x + u_2\partial_y + u_3\partial_z)u$

$u_1\partial_x u + u_2\partial_y u + u_3\partial_z u$ $:\mathbb{R}^3\to\mathbb{R}^3$

$\partial_x u = \partial_x(\langle u_1,u_2,u_3\rangle)$
$= \langle\partial_x u_1, \partial_x u_2, \partial_x u_3\rangle$

operators on vectors

$Curl\ F = \nabla \times F = \langle \partial_x, \partial_y, \partial_z \rangle \times F$ $:\mathbb{R}^3\to\mathbb{R}^3$

$Div\ F = \nabla \cdot F = \langle \partial_x, \partial_y, \partial_z \rangle \cdot F$ $:\mathbb{R}^3\to\mathbb{R}^1$

$$\nabla\times F = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\times\begin{pmatrix}F_1\\ F_2\\ F_3\end{pmatrix} = \begin{pmatrix}\partial_y F_3 - \partial_z F_2\\ \vdots\\ \vdots\end{pmatrix}$$

$$\nabla\cdot F = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$F = \langle \ln(y) + 2xy^3,\ 3x^2y^2 + \frac{x}{y}\rangle$

is F conservative?

$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$? $\frac{\partial F_2}{\partial x} = 6xy^2 + \frac{1}{y}$

$\frac{\partial F_1}{\partial y} = \frac{1}{y} + 6xy^2$ yes.

Find potential

if $\nabla f = F$

$\Rightarrow$ $f_x = \ln(y) + 2xy^3$

int. wrt x $f = \ln(y)x + x^2y^3 + g(y)$ (why?)

now $f_y = \frac{x}{y} + 3x^2y^2 + g'(y) = 3x^2y^2 + \frac{x}{y}$

$g'(y) = 0$

$\Rightarrow g(y) = c$

$f = \ln(y)x + x^2y^3 + g(y)$

f $\uparrow\int dx$ $\frac{\partial}{\partial y}$

$F = \langle F_1, F_2, F_3\rangle$ find f s.t. $\nabla f = F$

$f_x = F_1 \Rightarrow f = \int F_1\,dx + g(y,z)$

$\frac{\partial}{\partial y} f = F_2$ & find g.

Theorem:

1. $Curl(\nabla f) = 0$
2. $curl(F) = 0$ implies F is conservative (on a simply connected set)
3. Div(Curl(F))=0

Exercises:

1. Compute curl and divergence of
    a. $xyz\boldsymbol{i} - x^2y\boldsymbol{k}$
    b. $\cos xz\,\boldsymbol{j} - \sin xy\,\boldsymbol{k}$
2. Determine whether or not the vector field is conservative, if it is, find its potential
    a. $\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ $f = xyz^3 + c$
    b. $\langle 2xy, (x^2+2yz), y^2\rangle$
    c. $\langle ye^{-x}, e^{-x}, 2z\rangle$
3. For $\vec{r} = \langle x, y, z\rangle$ verify the following $|\vec{r}| = r$
    a. $\nabla\cdot\vec{r} = 3$
    b. $\nabla\cdot(r\vec{r}) = 4r$
    c. $\nabla r = \frac{\vec{r}}{r}$
    d. $\nabla\times\vec{r} = 0$
4. Show that if f is harmonic, then $\oint_C \nabla f\cdot\vec{n}ds = 0$

    Use $\oint_C \vec{F}\cdot\vec{n}ds = \iint_D div\,\vec{F}(x,y)dA$

$\nabla f(x,y) = (f_x, f_y)$

$f(r,\theta)$, $f_x = f_r r_x + f_\theta \theta_x$

$f_y = f_r r_y + f_\theta \theta_y$

div(grad(f))

$\nabla\cdot(\nabla f) = 0$

with formula, let $F = \nabla f$

$\text{div}\,F = \nabla\cdot\nabla f = 0$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}\cdot\vec{n}(r(t))\cdot|r'(t)|\,dt$$

$r = x(t)i + y(t)j$ $T(t) = \frac{x'}{|r'(t)|}\hat{i} + \frac{y'}{|r'(t)|}\hat{j}$

$$n(t) = \frac{y'}{|r'(t)|}\hat{i} - \frac{x'}{|r'(t)|}\hat{j}$$

$$\int_a^b\left(\frac{F_1\cdot y'}{|r'(t)|} - \frac{F_2\,x'}{|r'(t)|}\right)|r'(t)|\,dt$$

$$\int_a \left(\frac{F_1 \cdot y'}{|r'(t)|} - \frac{F_2 x'}{|r'(t)|} \right) |r'(t)| \, dt$$

$$\int_a^b F_1 y' - F_2 x' \quad dt$$

$$\int P \, dx + Q \, dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\underbrace{\int_C F_1 \, dy - F_2 \, dx = \iint \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \, dA}= \iint \nabla \cdot F \quad dA$$